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Propagator for a charged anisotropic oscillator in a constant magnetic field

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Abstract. The propagator for a charged, anisotropic harmonic oscillator in a constant magnetic field is computed. Making use of this an explicit expression is given for the path integral to an action with generalised memory recently considered in the literature.

1. Introduction

The action of a charged, anisotropic harmonic oscillator in a constant magnetic field is

$$S[\boldsymbol{x}]=\int_0^\beta \mathrm{d}t\,\tfrac{1}{2}m[(\dot{x}^2+\dot{y}^2+\dot{z}^2)-(\omega_x^2x^2+\omega_y^2y^2+\omega_z^2z^2)+\omega(x\dot{y}-y\dot{x})]. \qquad (1)$$

Here m denotes the mass, ω_x, ω_y, and ω_z are the oscillator frequencies along the x, y, and z directions, and ω is the cyclotron frequency. The external constant magnetic field has been chosen to point in the z direction. The propagator is given by the path integral

$$K(\boldsymbol{x}_\beta, \boldsymbol{x}_0; \beta)=\int_{\boldsymbol{x}_0}^{\boldsymbol{x}_\beta} \mathscr{D}\boldsymbol{x}\, \exp\{(\mathrm{i}/\hbar)S[\boldsymbol{x}]\}. \qquad (2)$$

The action (1) describes any self-oscillatory charged system in the limit of small oscillations, which is subjected to an attractive directionally dependent force and to a constant magnetic field. To give an example, consider the atomic truncs of an anisotropic metal lattice in an external constant magnetic field. Thus various applications are at hand. As far as we know, the propagator (2) has not so far been computed. Cheng (1984) finds a partial result and he also shows that (2) is related to the propagator of a one-dimensional time-dependent forced oscillator with generalised memory. This is achieved by performing the integrations with respect to the z and x coordinates. We are going to evaluate (2) directly extending the results of Cheng (1984). In particular an explicit expression of the above mentioned memory integral easily follows, see remark. Special cases of (2) are treated in the literature. In particular, see Jones and Papadopoulos (1971) for the isotropic case $\omega_x=\omega_y=\omega_z$.

2. Computation of the propagator

Since (1) is quadratic, the propagator (2) factorises:

$$K(\boldsymbol{x}_\beta, \boldsymbol{x}_0; \beta) = F(\beta)\exp\{(\mathrm{i}/\hbar)S[\bar{\boldsymbol{x}}]\} \tag{3}$$

cf Papadopoulos (1978), where $S[\bar{\boldsymbol{x}}]$ is the classical (stationary) action along the classical path $\bar{\boldsymbol{x}}$ and $F(\beta)$ denotes the normalisation factor. Since the original problem does not contain a memory term the latter follows from $S[\bar{\boldsymbol{x}}]$ computing the Van Vleck-Morette determinant

$$F(\beta) = \left\{\det\left[\frac{1}{2\pi\mathrm{i}\hbar}\left(\frac{\partial^2 S[\bar{\boldsymbol{x}}]}{\partial \boldsymbol{x}_\beta \partial \boldsymbol{x}_0}\right)\right]\right\}^{1/2}. \tag{4}$$

The equations of motion determining the classical path are

$$\ddot{x} + \omega_x^2 x = +\omega\dot{y}, \qquad \ddot{y} + \omega_y^2 y = -\omega\dot{x}, \qquad \ddot{z} + \omega_z^2 z = 0. \tag{5}$$

From the characteristic polynomials one gets the frequencies ω_z for the z coordinate and

$$\lambda_\pm^2 = \tfrac{1}{2}(\omega^2 + \omega_x^2 + \omega_y^2) \pm \tfrac{1}{2}[(\omega^2 + \omega_x^2 + \omega_y^2)^2 - 4\omega_x^2\omega_y^2]^{1/2} \tag{6}$$

for the coupled x and y coordinates. Thus $z(t)$ is a linear combination of $\sin(\omega_z t)$ and $\cos(\omega_z t)$, and $x(t)$ and $y(t)$ are linear combinations of $\sin(\lambda_\pm t)$ and $\cos(\lambda_\pm t)$. They are fixed by the boundary conditions $\boldsymbol{x}_0 = \boldsymbol{x}(0)$ and $\boldsymbol{x}_\beta = \boldsymbol{x}(\beta)$.

Making use of the equations of motion one sees that

$$S[\bar{\boldsymbol{x}}] = \tfrac{1}{2}m(\bar{\boldsymbol{x}}_\beta\dot{\bar{\boldsymbol{x}}}_\beta - \bar{\boldsymbol{x}}_0\dot{\bar{\boldsymbol{x}}}_0). \tag{7}$$

Therefore,

$$S[\bar{\boldsymbol{x}}] = S[\bar{z}] + S[\bar{x}, \bar{y}] \tag{8}$$

where

$$S[\bar{z}] = \tfrac{1}{2}m[\omega_z/\sin(\omega_z\beta)][(z_\beta^2 + z_0^2)\cos(\omega_z\beta) - 2z_0 z_\beta] \tag{9}$$

and, by some lengthy algebraic manipulations,

$$\begin{aligned} S[\bar{x}, \bar{y}] = (m/2D)[&(x_\beta^2 + x_0^2)a_1 + (y_\beta^2 + y_0^2)a_2 + x_0 x_\beta a_3 + y_0 y_\beta a_4 \\ &+ (x_0 y_\beta - x_\beta y_0)a_5 + (x_\beta y_\beta - x_0 y_0)a_6], \end{aligned} \tag{10}$$

$$\begin{aligned} D = {}& 2 - 2\cos(\lambda_+\beta)\cos(\lambda_-\beta) + \{[(\omega_x^2 - \omega_y^2)^2 + \omega^2(\omega_x^2 + \omega_y^2)]/\omega^2\omega_x\omega_y\} \\ &\times \sin(\lambda_+\beta)\sin(\lambda_-\beta), \\ a_1 = {}& [(\lambda_+^2 - \lambda_-^2)/\omega^2\omega_y][(\lambda_+\omega_x - \lambda_-\omega_y)\cos(\lambda_+\beta)\sin(\lambda_-\beta) \\ &+ (\lambda_+\omega_y - \lambda_-\omega_x)\cos(\lambda_-\beta)\sin(\lambda_+\beta)], \\ a_2 = {}& [(\lambda_+^2 - \lambda_-^2)/\omega^2\omega_x][(\lambda_+\omega_y - \lambda_-\omega_x)\cos(\lambda_+\beta)\sin(\lambda_-\beta) \\ &+ (\lambda_+\omega_x - \lambda_-\omega_y)\cos(\lambda_-\beta)\sin(\lambda_+\beta)], \\ a_3 = {}& (2/\omega^2\omega_y)(\lambda_+^2 - \lambda_-^2)[(\lambda_-\omega_x - \lambda_+\omega_y)\sin(\lambda_+\beta) - (\lambda_+\omega_x - \lambda_-\omega_y)\sin(\lambda_-\beta)], \\ a_4 = {}& (2/\omega^2\omega_x)(\lambda_+^2 - \lambda_-^2)[(\lambda_-\omega_y - \lambda_+\omega_x)\sin(\lambda_+\beta) - (\lambda_+\omega_y - \lambda_-\omega_x)\sin(\lambda_-\beta)], \\ a_5 = {}& (2/\omega)(\lambda_+^2 - \lambda_-^2)[\cos(\lambda_-\beta) - \cos(\lambda_+\beta)], \end{aligned}$$

$$a_6 = [(\omega_x^2 - \omega_y^2)/\omega]\{2 - [(\omega^2 + \omega_x^2 + \omega_y^2)/\omega_x\omega_y] \times \sin(\lambda_+\beta)\sin(\lambda_-\beta) - 2\cos(\lambda_+\beta)\cos(\lambda_-\beta)\}.$$

Hence, by (4), the normalisation factor follows

$$F(\beta) = \left(\frac{m\omega_z}{2\pi i\hbar\sin(\omega_z\beta)}\right)^{1/2}\frac{m}{2\pi i\hbar}\frac{1}{\omega D^{1/2}}(\lambda_+^2 - \lambda_-^2). \tag{11}$$

Combining (8)-(11) the computation of the propagator according to (3) is accomplished.

Remark. According to Cheng (1984) one gets

$$K(\boldsymbol{x}_\beta, \boldsymbol{x}_0; \beta) = K_{\omega_x}(x_\beta, x_0; \beta)K_{\omega_z}(z_\beta, z_0; \beta)\exp[(im\omega/2\hbar)(x_\beta y_\beta - x_0 y_0)]H(y_\beta, y_0; \beta) \tag{12}$$

where

$$K_{\omega_x}(x_\beta, x_0; \beta) = \left(\frac{m\omega_x}{2\pi i\hbar\sin(\omega_x\beta)}\right)^{1/2}\exp\left(\frac{im\omega_x}{2\hbar\sin(\omega_x\beta)}[(x_\beta^2 + x_0^2)\cos(\omega_x\beta) - 2x_0x_\beta]\right),$$

$$K_{\omega_z}(z_\beta, z_0; \beta) = \left(\frac{m\omega_z}{2\pi i\hbar\sin(\omega_z\beta)}\right)^{1/2}\exp\left(\frac{im\omega_z}{2\hbar\sin(\omega_z\beta)}[(z_\beta^2 + z_0^2)\cos(\omega_z\beta) - 2z_0z_\beta]\right),$$

and the path integral

$$H(y_\beta, y_0; \beta) = \int_{y_0}^{y_\beta}\mathscr{D}y\exp\left(\frac{im}{2\hbar}\int_0^\beta dt[\dot{y}^2 - (\omega^2 + \omega_y^2)y^2 + \frac{2}{m}f(t) + M(t)]\right) \tag{13}$$

with the external force

$$f(t) = [m\omega\omega_x/\sin(\omega_x\beta)][x_0\cos\omega_x(\beta - t) - x_\beta\cos(\omega_x t)] \tag{14}$$

and the memory potential

$$M(t) = [2\omega^2\omega_x/\sin(\omega_x\beta)]y(t)\cos[\omega_x(\beta - t)]\int_0^t ds\, y(s)\cos(\omega_x s). \tag{15}$$

Comparing (3) with (12) one obtains

$$H(y_\beta, y_0; \beta) = \frac{\exp[-(im\omega/2\hbar)(x_\beta y_\beta - x_0y_0)]}{K_{\omega_x}(x_\beta, x_0; \beta)} \times\frac{m}{2\pi i\hbar}\frac{1}{wD^{1/2}}(\lambda_+^2 - \lambda_-^2)\exp\{(i/\hbar)S[\bar{x}, \bar{y}]\}. \tag{16}$$

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